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ORIGINAL PAPER

The Betti reciprocity principle and the normal boundary component control problem for linear elastic systems

David L. Russell

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Abstract The background for this article is the question of modification of the geometric configuration of an elastic structure by means of "volume" type actuation. In this actuation mode stresses are applied to the elastic body by injection/extraction of a fluid into, or from, a large number of vacuoles in the elastic "matrix" material. Previous articles by the author, and others, have examined this process and studied its effectiveness in the context of a "naive" continuous model. The present paper continues along these lines, exploring "normal boundary component controllability" criterion for determining achievable configurations for the controlled system in the two-dimensional case. Connections with conformal mapping lead to affirmative results for approximate controllability in this sense and Fourier series techniques provide exact controllability results for the case wherein the domain of the uncontrolled system is a two-dimensional disk.

Keywords Linear elasticity · Control of elastic systems · Volume type control · Betti reciprocity principle

AMS-MOS Classifications 74B05 · 74G10 · 74G75 · 93C55

1 Model formulation: control objectives

In static control of elastic systems, as discussed earlier in [5–7] and elsewhere, one is concerned with the controlled modification of the geometric configuration, i.e., the "shape," of an elastic body by means of attached or embedded actuators. These are idealized as being continuously distributed throughout all or part of the elastic body, applying a stress distribution determined by externally supplied control signals. In the

D. L. Russell (✉)
Department of Mathematics,
Virginia Polytechnic Institute and State University,
Blacksburg, VA 24061-123, USA
e-mail: russell@math.vt.edu

present paper, we consider an isotropic linear elastic solid in two-dimensional space R^2. The undeformed body occupies a region $\mathcal{R}_0$ in R^2 with smooth boundary $\mathcal{B}_0$. We denote vectors (points) in R^2 by capital Latin letters; viz: X; in terms of coordinates, $X = (x, y)$. The unit exterior (with respect to $\mathcal{R}_0$) normal at a point $X \in \mathcal{B}_0$ will be denoted by $N(X)$. Deformation of the elastic body is described by means of a *deformation map*

$$\mathcal{X}(X) \equiv X + \Xi(X),$$

where $\Xi(X)$ is the *displacement field* with components ξ, η. The map $\mathcal{X}$ carries the region $\mathcal{R}_0$ into a new region $\mathcal{R}$ and is assumed, at the least, to belong to $H^1(\mathcal{R}_0)$. Further, $\mathcal{X}$ is assumed bijective with positive Jacobian determinant throughout $\mathcal{R}_0$.

In the present article we continue to use the "naive," model developed in the references cited above. Some steps toward a more sophisticated theory, obtained through homogenization studies, have been taken in a later paper [8]. A major strength of the naive model lies in the fact that it allows most of the standard control and observation theory to have readily identified counterparts in elastic system control. Its major shortcoming lies in the fact that, in implementation, the small vacuoles to which the pressure $p(x)$ is applied would expand or contract and the elastic material between the vacuoles would undergo local compression or expansion. The naive model ignores these localized strains and the potential energy thus stored.

Our objective here is to study "volume actuation" of linear elastic systems. This actuation mode envisions isotropic stresses applied to the elastic body through injection or extraction of a fluid into or out of, respectively, a large number of small vacuoles distributed throughout the body, creating thereby a pressure distribution $p(X),\ X \in \mathcal{R}_0$. In the naive model, the Hamiltonian minimized by the displacement field $\Xi(X)$ with components $\xi(x, y), \eta(x, y)$ is

$$\frac{1}{2}\int\int_{\mathcal{R}_0}\left[(\lambda+\nu)\left(\frac{\partial\xi}{\partial x}+\frac{\partial\eta}{\partial y}\right)^2+\nu\left(\frac{\partial\xi}{\partial x}-\frac{\partial\eta}{\partial y}\right)^2 +\nu\left(\frac{\partial\xi}{\partial y}+\frac{\partial\eta}{\partial x}\right)^2-2p\left(\frac{\partial\xi}{\partial x}+\frac{\partial\eta}{\partial y}\right)\right]\mathrm{d}X, \tag{1.1}$$

where λ and ν are the Lamé constants [9]. The existence of a unique, modulo uniform translation and infinitesimal rotation, minimizing $\Xi(X)$ for given $p \in L^2(\mathcal{R}_0)$ is a consequence of the Lax-Milgram theorem [11] (Chapter III) and the coercivity implied by Korn's inequalities [2,10]. The result extends to $F \in H^{-1}(\mathcal{R}_0)$ and $G \in H^{-\frac{1}{2}+\epsilon}(\mathcal{B}_0),\ \epsilon > 0$, using the *trace theorem* [3]. In the articles cited above, we have developed the necessary conditions for minimization in terms of a system of equilibrium partial differential equations and boundary conditions.

We will review several modes of controllability which might plausibly be formulated in the present context.

- *Full domain control*: This mode of control seeks to specify the whole mapping $\mathcal{X}(X) : \mathcal{R}_0 \hookrightarrow \mathcal{R}$ by means of the control $p(X)$. This involves more than the "shape" of the new geometric configuration $\mathcal{R}$; it also seeks to specify the displacements in the interior of $\mathcal{R}_0$ as well. In most applications this control mode cannot be realized; we do not pursue it further here.
- *Boundary image control*: Clearly the controlled geometry of the elastic body can be specified by giving the *boundary* $\mathcal{B}$ of the image region $\mathcal{R}$ and requiring

$\mathcal{X}(X) \in \mathcal{B},\ X \in \mathcal{B}_0$. The solution of this problem is typically not unique and one introduces optimality criteria to select the best control/actuator configurations. This is ordinarily not a linear problem even when posed for a linear elastic system. It has been studied extensively in [5].

- *Normal boundary component control*: In this control mode a scalar function $h(X),\ X \in \mathcal{R}_0$, is given in an appropriate space and, with $N(X),\ X \in \mathcal{B}_0$, denoting the unit exterior normal vector, one requires that the controlled displacement $\Xi(X)$ should satisfy $N(X)^*\Xi(X) = h(X),\ X \in \mathcal{B}_0$. This has the advantage that the overall problem remains linear. The disadvantage is that the relationship between $h(X)$ and the new boundary $\mathcal{B}$ (hence the new "shape") is typically less than obvious.

In the present article, we study normal boundary component control, via volume actuation with control pressure $p(X),\ X \in \mathcal{R}_0 \subset R^2$. We begin by studying approximate controllability in this context, relating it to the familiar *Betti Reciprocity Principle* of linear elasticity. Then we study the corresponding exact controllability problem in the case where $\mathcal{R}_0$ is the unit disk, $\mathcal{B}_0$ the unit circle, in R^2, including resolution of the relationship between $h(X)$ and the new boundary $\mathcal{B}$ when $\mathcal{B}$ is close to the unit circle $\mathcal{B}_0$ in a particular sense.

2 Approximate normal boundary component controllability with volume actuation in R^2

In the naive model of volume control with pressure distribution $p(X),\ X \in \mathcal{R}_0 \in R^2$, the Hamiltonian minimized by the displacement field $\Xi(X)$ with components $\xi(x,y), \eta(x,y)$ is

$$\frac{1}{2}\int\int_{\mathcal{R}_0}\left[(\lambda+\nu)\left(\frac{\partial\xi}{\partial x}+\frac{\partial\eta}{\partial y}\right)^2+\nu\left(\frac{\partial\xi}{\partial x}-\frac{\partial\eta}{\partial y}\right)^2 + \nu\left(\frac{\partial\xi}{\partial y}+\frac{\partial\eta}{\partial x}\right)^2 - 2p\left(\frac{\partial\xi}{\partial x}+\frac{\partial\eta}{\partial y}\right)\right]\mathrm{d}X. \tag{2.1}$$

We specify a boundary component control problem by requiring

$$N(X)^*\Xi(X) = h(x),\ X \in \mathcal{B}_0, \tag{2.2}$$

where $h(x)$ is a scalar function in $H^{(1-\epsilon)/2}(\mathcal{B}_0) \subset L^2(\mathcal{B}_0)$ for small positive ϵ. We will refer to this as the *primal* (*control*) *problem*. The equilibrium displacement $\Xi(X)$ in the presence of the pressure distribution $p(X)$ satisfies, for an arbitrary $\delta\Xi = (\delta\xi,\delta\eta) \in H^1(\mathcal{R}_0)$, the variational equation

$$\int\int_{\mathcal{R}_0}\left[(\lambda+\nu)\left(\frac{\partial\xi}{\partial x}+\frac{\partial\eta}{\partial y}\right)\left(\frac{\partial\delta\xi}{\partial x}+\frac{\partial\delta\eta}{\partial y}\right)+\nu\left(\frac{\partial\xi}{\partial x}-\frac{\partial\eta}{\partial y}\right)\left(\frac{\partial\delta\xi}{\partial x}-\frac{\partial\delta\eta}{\partial y}\right) +\nu\left(\frac{\partial\xi}{\partial y}+\frac{\partial\eta}{\partial x}\right)\left(\frac{\partial\delta\xi}{\partial y}+\frac{\partial\delta\eta}{\partial x}\right) - p\left(\frac{\partial\delta\xi}{\partial x}+\frac{\partial\delta\eta}{\partial y}\right)\right]\mathrm{d}X = 0. \tag{2.3}$$

Together with the primal problem we consider a *dual* (*observation*) *problem* for the related *dual system* in $\mathcal{R}_0 \subset R^2$ with smooth boundary $\mathcal{B}_0$. The latter is a second linear elastic system with displacement field $\Omega(X) = \binom{u(X)}{v(X)}$. In the presence of a normally oriented boundary force $g(X) \in L^2(\mathcal{B}_0)$, the dual Hamiltonian takes the form

$$\mathcal{L} = \frac{1}{2}\int_{\mathcal{R}_0}\left[(\lambda+\nu)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)^2 + \nu\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)^2 + \nu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\right]\mathrm{d}X \\ - \int_{\mathcal{B}_0} g(X)N(X)^*\Omega(X)\,\mathrm{d}s, \tag{2.4}$$

The dual displacement $\Omega = \binom{u}{v}$ minimizing this Hamiltonian with respect to general displacements in $H^1(\mathcal{R}_0)$ then satisfies, for an arbitrary $\delta\Omega = (\delta u, \delta v) \in H^1(\mathcal{R}_0)$,

$$\int_{\mathcal{R}_0}\left[\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\left(\frac{\partial \delta u}{\partial x}+\frac{\partial \delta v}{\partial y}\right) + \left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)\left(\frac{\partial \delta u}{\partial x}-\frac{\partial \delta v}{\partial y}\right)\right. \\ \left.+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\left(\frac{\partial \delta u}{\partial y}+\frac{\partial \delta v}{\partial x}\right)\right]\mathrm{d}X - \int_{\mathcal{B}_0} g(X)N(X)^*\delta\Omega(X)\,\mathrm{d}s = 0. \tag{2.5}$$

If in (2.3) we set $\delta\xi = u$, $\delta\eta = v$, u and v minimizing (2.4), and in (2.5) set $\delta u = \xi$, $\delta v = \eta$, ξ and η minimizing (2.1), and then compare the resulting two equations, we quickly obtain the version of the Betti Reciprocity Principle applying to this case:

$$\int\int_{\mathcal{R}_0} p(X)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)(X)\,\mathrm{d}X = \int_{\mathcal{B}_0} g(X)N(X)^*\Xi(X)\,\mathrm{d}s. \tag{2.6}$$

If the normal boundary displacement components $N(X)^*\Xi(X) = h(X)$ achievable via the primal system via controls $p \in L^2(\mathcal{R}_0)$ are not dense in $L^2(\mathcal{B}_0)$, then there will be a function $g \in L^2(\mathcal{B}_0)$ for which the left-hand side of (2.6) is zero for all $p \in L^2(\mathcal{R}_0)$. But this clearly implies $\nabla^*\Omega(X) \equiv \left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)(X) \equiv 0,\ X \in \mathcal{R}_0$. Thus we have the

Control/observation duality result: *The primal control system fails to have the property of approximate normal boundary component controllability in $L^2(\mathcal{B}_0)$ if and only if the dual observation system fails to be observable (detectable) in the sense that there is an input $g \in L^2(\mathcal{B}_0)$ for which the "observation" divergence $\nabla^*\Omega(X) \equiv 0,\ X \in \mathcal{R}_0$.*

3 Approximate normal boundary component controllability for a general region $\subset R^2$

Our objective in this section is to prove the following theorem.

Theorem 3.1 *Let $\mathcal{R}_0$ be a simply connected domain in R^2 with smooth boundary $\mathcal{B}_0$ and unit exterior normal $N(X) = \binom{n_1(x,y)}{n_2(x,y)}$, $X \in \mathcal{B}_0$. Then the primal system has the property of approximate normal boundary component controllability.*

Proof We will give a proof by contradiction. Let us suppose there is a function $g(X) \in L^2(\mathcal{B}_0)$ such that $\nabla^*\Omega \equiv \left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)(X) \equiv 0$ in $\mathcal{R}_0$. Then, redefining g/ν as g again, (2.5) must take the form

$$\int_{\mathcal{R}_0} \left[\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\left(\frac{\partial \delta u}{\partial x} - \frac{\partial \delta v}{\partial y}\right) + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\left(\frac{\partial \delta u}{\partial y} + \frac{\partial \delta v}{\partial x}\right)\right] \mathrm{d}X$$
$$- \int_{\mathcal{B}_0} g(X) N(X)^* \delta\Omega(X)\, \mathrm{d}s = 0. \tag{3.1}$$

It is well known (cf. e.g. [11]) that the minimizer of (2.4) must, in fact, lie in $H^2(\mathcal{R}_0)$ with first order partial derivatives having $H^{(1-\epsilon)/2}$ traces on $\mathcal{B}_0$, $\epsilon > 0$. Thus we can see, using the divergence theorem, that (3.1) reduces to

$$0 = \int_{\mathcal{B}_0} \left[\begin{pmatrix} \delta u \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \delta v \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ \delta u \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - \delta v \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) \end{pmatrix} \cdot \begin{pmatrix} n_1(X) \\ n_2(X) \end{pmatrix} - g(X) N(X)^* \delta\Omega(X)\right] \mathrm{d}s$$
$$- \int_{\mathcal{R}_0} \left[\delta u \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \delta v \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right.$$
$$\left. + \delta u \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - \delta v \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\right] \mathrm{d}X.$$

Setting the coefficients of δu and δv equal to zero separately on $\mathcal{R}_0$ and $\mathcal{B}_0$ we obtain the system of partial differential equations

$$\begin{aligned} \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) &= 0, \\ \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) &= 0, \end{aligned} \qquad X \in \mathcal{R}_0 \tag{3.2}$$

and the boundary conditions

$$\begin{aligned} \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) n_1 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) n_2 &= g n_1, \\ \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) n_1 - \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) n_2 &= g n_2, \end{aligned} \qquad X \in \mathcal{B}_0. \tag{3.3}$$

Now, in fact, Eq. (3.2) reduce to $\Delta u = 0$, $\Delta v = 0$, where Δ is the Laplacian. So u and v are harmonic in $\mathcal{R}_0$. On the other hand, denoting the exterior normal derivative on $\mathcal{B}_0$ by $\frac{\partial}{\partial N}$ and the tangential derivative in the mathematically positive direction on (the curve) $\mathcal{B}_0$ by $\frac{\partial}{\partial T}$, (3.3) gives

$$\begin{aligned} \frac{\partial u}{\partial N} - \frac{\partial v}{\partial T} &= g n_1, \\ \frac{\partial u}{\partial T} + \frac{\partial v}{\partial N} &= g n_2, \end{aligned} \qquad X \in \mathcal{B}_0. \tag{3.4}$$

The lemma to follow shows that if (3.4) is satisfied with u and v harmonic in $\mathcal{R}_0$, then $g = 0$ in $L^2(\mathcal{B}_0)$. The proof of the theorem then follows from the control/observation duality result set forth previously.

Lemma 3.1 *Let $u(x, y)$, $v(x, y)$ be harmonic functions in the simply connected domain $H^2(\mathcal{R}_o)$ with smooth boundary $\mathcal{B}_0$. Suppose there is a real valued function $g(x, y)$ (minimally) in $L^2(\mathcal{B}_0)$ such that (3.4) is satisfied on $\mathcal{B}_0$. Then $g = 0$ in $L^2(\mathcal{B}_0)$ and the complex function $u(x, y) + i\,v(x, y)$ is holomorphic in $\mathcal{R}_0$.*

Proof It is readily verified that, with $z = x + i\,y$, we may represent $u(x, y) + i\,v(x, y)$ in the form,

$$u(x, y) + i\,v(x, y) = \phi(z) + \psi(\overline{z}),$$

where $\phi(z)$ and $\psi(z)$ are holomorphic in $\mathcal{R}_0$ and the conjugate domain $\overline{\mathcal{R}}_0$, respectively. Writing

$$\phi(z) = \phi_1(x,y) + i\phi_2(x,y), \quad z \in \mathcal{R}_0, \qquad \psi(z) = \psi_1(x,y) + i\psi_2(x,y), \ z \in \overline{\mathcal{R}}_0$$

then for $z = x + iy \in \mathcal{R}_0$ we have, using the Cauchy-Riemann equations,

$$\psi'(\overline{z}) = \frac{\partial \psi_1}{\partial x}(x,-y) + i\frac{\partial \psi_2}{\partial x}(x,-y) = \frac{\partial \psi_1}{\partial x}(x,-y) - i\frac{\partial \psi_1}{\partial y}(x,-y).$$

Elementary computations show, for $z = x + iy \in \mathcal{R}$, that

$$\frac{\partial \phi_1}{\partial x}(x,-y) = \frac{1}{2}\left(\frac{\partial u}{\partial x}(x,y) - \frac{\partial v}{\partial y}(x,y)\right), \quad \frac{\partial \psi_1}{\partial y}(x,-y) = \frac{1}{2}\left(\frac{\partial u}{\partial y}(x,y) + \frac{\partial v}{\partial x}(x,y)\right).$$

Thus, for $z = (x,y) \in \mathcal{B}$, application of (3.4) yields

$$\begin{aligned}
\psi'(\overline{z}) &= \frac{1}{2}\left(\frac{\partial u}{\partial N}n_1 - \frac{\partial u}{\partial T}n_2 - \frac{\partial v}{\partial N}n_2 - \frac{\partial v}{\partial T}n_1\right) + \frac{i}{2}\left(\frac{\partial u}{\partial N}n_2 + \frac{\partial u}{\partial T}n_1 + \frac{\partial v}{\partial N}n_1 - \frac{\partial v}{\partial T}n_2\right) \\
&= \frac{1}{2}\left(n_1\left(\frac{\partial u}{\partial N} - \frac{\partial v}{\partial T}\right) - n_2\left(\frac{\partial u}{\partial T} + \frac{\partial v}{\partial N}\right)\right) + \frac{i}{2}\left(n_1\left(\frac{\partial u}{\partial T} + \frac{\partial v}{\partial N}\right) + n_2\left(\frac{\partial u}{\partial N} - \frac{\partial v}{\partial T}\right)\right) \\
&= \frac{1}{2}\left(n_1^2 g - n_2^2 g\right) + \frac{i}{2}\left(2n_1 n_2 g\right) = \frac{1}{2}\left(n_1 + in_2\right)^2 g. \qquad (3.5)
\end{aligned}$$

Now let $z = \chi(\zeta)$ map the unit disk $|\zeta| \leq 1$ onto $\mathcal{R}_0 \cup \mathcal{B}_0$. Since $\mathcal{B}_0$ is smooth the mapping is continuously differentiable and conformal on the closed domain [1,4]. Then the function

$$f(z) \equiv \chi'(\zeta)\zeta = \chi'\left(\chi^{-1}(z)\right)\chi^{-1}(z)$$

is analytic in $\mathcal{R}_0$ and continuous on $\mathcal{R}_0 \cup \mathcal{B}_0$. Since $\zeta = e^{i\omega}$, considered as a two-dimensional vector, is the unit exterior normal to the unit disk at the point ζ, conformality shows that $\chi'(\zeta)\zeta$, considered as a vector, must be a positive multiple of the unit exterior normal to $\mathcal{R}_0$ at $z = \chi(\zeta)$. That is, for some positive $q = q(z)$,

$$\begin{aligned}
f(z) &\equiv q(z)\left(n_1(z) + in_2(z)\right), \qquad z \in \mathcal{B}_0 \ \Rightarrow \quad f(z)^2 \\
&= q(z)^2\left(n_1(z) + in_2(z)\right)^2, \quad z \in \mathcal{B}_0.
\end{aligned}$$

Then, from (3.5),

$$\psi'(\overline{z}) = \frac{g(z)}{q(z)^2}f(z)^2 \ \Rightarrow \ \psi'(\overline{z})\overline{f(z)}^2 = \frac{g(z)}{q(z)^2}\left|f(z)\right|^4, \quad z \in \mathcal{B}_0.$$

Conjugating, the function

$$h(z) = \overline{(\psi'(\overline{z}))}f(z)^2$$

is analytic in $\mathcal{R}_0$, continuous on $\mathcal{R}_0 \cup \mathcal{B}_0$ and real valued on $\mathcal{B}_0$, which implies $h(z)$ is a constant. Since $f(\chi(0)) = 0$, we must have

$$h(z) \equiv 0, \quad z \in \mathcal{R}_0 \cup \mathcal{B}_0 \ \Rightarrow \ g(z) \equiv 0, \quad z \in \mathcal{B}_0.$$

Since $f(z) \neq 0$ on $\mathcal{B}_0$, we conclude $\psi'(\overline{z}) \equiv 0$, $z \in \mathcal{B}_0$. Then $\psi(\overline{z}) \equiv c$, c constant, in $\mathcal{R}_0 \cup \mathcal{B}_0$ and $u(x,y) + iv(x,y) \equiv \phi(z) + ic$ is holomorphic in $\mathcal{R}_0 \cup \mathcal{B}_0$. The proof is complete.

Remark To show that the smoothness assumption on $\mathcal{B}_0$ is necessary we consider the case where $\mathcal{R}_0$ is a rectangle and $u(x,y) \equiv x$, $v(x,y) \equiv -y$. Here the obvious counterparts of Theorem 3.1 and Lemma 3.1 fail because, as follows from the Schwarz-Christoffel formula, $\chi'(\zeta) \equiv \chi'\left(\chi^{-1}(z)\right)$ assumes arbitrarily large values as z approaches any one of the corners of the rectangle. Then $h(z)$ behaves similarly and the maximum principle cannot be applied to show that $\mathrm{Im}((h(z)))$ vanishes in $\mathcal{R}_0$, upon which the proof of $h(z)$ constant depends.

Expanding further upon the point just made, we can identify a large number, indeed, infinitely many, domains $\mathcal{R}_0$ with *piecewise smooth* boundaries $\mathcal{B}_0$ for which the counterpart of Lemma 3.1 does not obtain and, consequently, one does not have approximate normal boundary component controllability with volume actuation as in Theorem 3.1. Let $U(x,y) + i\,V(x,y)$ be an analytic function in a domain $\hat{\mathcal{D}} \in R^2$ and, in the conjugate domain $\mathcal{D} = \overline{\hat{\mathcal{D}}}$ define

$$u(x,y) = U(x,-y), \quad v(x,y) = V(x,-y).$$

Clearly $u(x,y)$, $v(x,y)$ are harmonic in $\mathcal{D}$ and, from the Cauchy-Riemann equations satisfied by $U(x,y)$, $V(x,y)$ in $\hat{\mathcal{D}}$, for $(x,y) \in \mathcal{D}$, we have

$$\mathrm{div}\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \frac{\partial}{\partial x}\left(U(x,-y)\right) + \frac{\partial}{\partial y}\left(V(x,-y)\right) = \frac{\partial U}{\partial x}(x,-y) - \frac{\partial V}{\partial y}(x,-y) = 0$$

and

$$\begin{aligned} \frac{\partial u}{\partial x}(x,y) - \frac{\partial v}{\partial y}(x,y) &= \frac{\partial}{\partial x}\left(U(x,-y)\right) - \frac{\partial}{\partial y}\left(V(x,-y)\right) \\ &= \frac{\partial U}{\partial x}(x,-y) + \frac{\partial V}{\partial y}(x,-y) = 2\,\frac{\partial U}{\partial x}(x,-y) = 2\,\frac{\partial u}{\partial x}(x,y), \end{aligned} \tag{3.6}$$

$$\begin{aligned} \frac{\partial u}{\partial y}(x,y) + \frac{\partial v}{\partial x}(x,y) &= \frac{\partial}{\partial y}\left(U(x,-y)\right) + \frac{\partial}{\partial x}\left(V(x,-y)\right) \\ &= -\frac{\partial U}{\partial y}(x,-y) + \frac{\partial V}{\partial x}(x,-y) = -2\,\frac{\partial U}{\partial y}(x,-y) = 2\,\frac{\partial u}{\partial y}(x,y). \end{aligned} \tag{3.7}$$

As a consequence of (3.6), (3.7), Eq. 3.3 becomes (for convenience redefining g as $2\,g$)

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial y} & -\frac{\partial u}{\partial x} \end{pmatrix}\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = g\begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \quad X \in \mathcal{B}. \tag{3.8}$$

Equation 3.8 has the form of an eigenvalue-eigenvector problem for the matrix appearing on the left. Computing

$$\det\begin{pmatrix} \frac{\partial u}{\partial x} - g & \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial y} & -\frac{\partial u}{\partial x} - g \end{pmatrix} = g^2 - \left(\frac{\partial u}{\partial x}^2 + \frac{\partial u}{\partial y}^2\right) \tag{3.9}$$

the eigenvalues are

$$g_+ = \sqrt{\frac{\partial u}{\partial x}^2 + \frac{\partial u}{\partial y}^2}, \quad g_- = -\sqrt{\frac{\partial u}{\partial x}^2 + \frac{\partial u}{\partial y}^2}. \tag{3.10}$$

The corresponding (orthogonal, since the matrix in (3.8) is symmetric) eigenvectors

$$N_+ = \begin{pmatrix} n_{1,+} \\ n_{2,+} \end{pmatrix}, \quad N_- = \begin{pmatrix} n_{1,-} \\ n_{2,-} \end{pmatrix}$$

must consequently be orthogonal to the first row of the matrix in (3.9) with g replaced by $g_\pm$, respectively, and hence to the vectors

$$T_+ = \begin{pmatrix} \frac{\partial u}{\partial x} - g_+ \\ \frac{\partial u}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} - \sqrt{\frac{\partial u}{\partial x}^2 + \frac{\partial u}{\partial y}^2} \\ \frac{\partial u}{\partial y} \end{pmatrix},$$
$$T_- = \begin{pmatrix} \frac{\partial u}{\partial x} - g_- \\ \frac{\partial u}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} + \sqrt{\frac{\partial u}{\partial x}^2 + \frac{\partial u}{\partial y}^2} \\ \frac{\partial u}{\partial y} \end{pmatrix}.$$

We may define a system of mutually orthogonal curves in $\mathcal{D}$ consisting of solutions of the two-dimensional $\left(X = \binom{x}{y}\right)$ systems

$$\mathcal{C}_+ : \; \frac{\mathrm{d}X_+}{\mathrm{d}t} = T_+(X_+), \qquad \mathcal{C}_- : \; \frac{\mathrm{d}X_-}{\mathrm{d}t} = T_-(X_-).$$

Any number of regions $\mathcal{R}_0$ may now be constructed by taking the boundary $\mathcal{B}_0$ to consist of alternating arcs taken from $\mathcal{C}_+,\ \mathcal{C}_-$, selected so that $\mathcal{B}_0$ forms one or more closed curves in $\mathcal{D}$. With the proper sign, $\pm N_+,\ \pm N_-$ on arcs in $\mathcal{C}_+,\ \mathcal{C}_-$, respectively, will be exterior normal vectors to $\mathcal{R}_0$ along those arcs, for which the equations (3.8), and hence (3.3), are satisfied for the respective values of $g_\pm$ as shown in (3.10).

In Fig. 1 below two regions $\mathcal{R}_0$ are shown, corresponding to $U(x,y) + i\,V(x,y) = (x + i\,y)^2,\ U(x,y) + i\,V(x,y) = (x + i\,y)^3$. Both curve systems $\mathcal{C}_+,\ \mathcal{C}_-$, are shown in the case at the left; only one at the right.

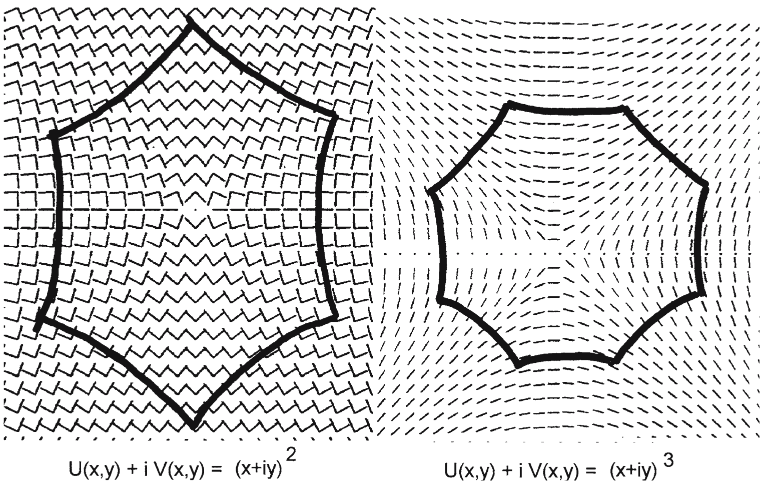

Fig. 1 Example regions for which Lemma 3.1 is not valid

4 Target region specification via normal boundary displacement component for the unit disk in R^2

For volume control with a pressure distribution $p(X),\ X \in \mathcal{R}_0$, the relevant Hamiltonian minimized by the displacement field $\Xi(X)$ with components $\xi(x,y), \eta(x,y)$ is

$$\frac{1}{2}\int\int_{\mathcal{R}_0}\left[(\lambda+\nu)\left(\frac{\partial\xi}{\partial x}+\frac{\partial\eta}{\partial y}\right)^2+\nu\left(\frac{\partial\xi}{\partial x}-\frac{\partial\eta}{\partial y}\right)^2 \right.$$
$$\left. +\nu\left(\frac{\partial\xi}{\partial y}+\frac{\partial\eta}{\partial x}\right)^2-2p\left(\frac{\partial\xi}{\partial x}+\frac{\partial\eta}{\partial y}\right)\right]\mathrm{d}X. \tag{4.1}$$

We specify a boundary component control problem by requiring

$$N(X)^*\Xi(X) = h(x), \quad X \in \mathcal{B}_0, \tag{4.2}$$

where $h(x)$ is a scalar function, at least of class C^1 on $\mathcal{B}_0$. Minor modifications to theorems given in [5] show that, assuming (4.2) can be realized, the control $p(X),\ X \in \mathcal{R}_0$ of least norm in $L^2(\mathcal{R}_0)$ realizing this objective takes the form $p(X) = \operatorname{div} U(X)$ for some solution $U(X)$ of the dual problem with $g(X)$ as described earlier and, as such, $p(X)$ is harmonic in $\mathcal{R}_0$. A further theorem in [5] then shows that the resulting displacement $\Xi(X)$ is such that the corresponding complex valued function $\xi(x,y) + i\,\eta(x,y)$ is analytic, and thus a conformal map. Application of the Cauchy-Riemann equations to $\xi(x,y),\ \eta(x,y)$ then shows that the controlled displacement in fact must minimize a reduced Hamiltonian

$$\frac{1}{2}\int\int_{\mathcal{R}_0}\left[(\lambda+\nu)\left(\frac{\partial\xi}{\partial x}+\frac{\partial\eta}{\partial y}\right)^2-2p\left(\frac{\partial\xi}{\partial x}+\frac{\partial\eta}{\partial y}\right)\right]\mathrm{d}X. \tag{4.3}$$

Restricting attention to the case where the domain $\mathcal{R}_0$ is the unit disk in R^2 and changing to polar coordinates with polar displacement coordinates σ and μ, we obtain the reduced Hamiltonian in polar form

$$\frac{1}{2}\int_0^{2\pi}\int_0^1\left[(\lambda+\nu)\left(\frac{\partial\sigma}{\partial r}+\frac{\sigma}{r}+\frac{1}{r}\frac{\partial\mu}{\partial\theta}\right)^2-2p\left(\frac{\partial\sigma}{\partial r}+\frac{\sigma}{r}+\frac{1}{r}\frac{\partial\mu}{\partial\theta}\right)\right]r\,\mathrm{d}r\,\mathrm{d}\theta. \tag{4.4}$$

The corresponding variational equation is then

$$\int_0^{2\pi}\int_0^1\left[(\lambda+\nu)\left(\frac{\partial\sigma}{\partial r}+\frac{\sigma}{r}+\frac{1}{r}\frac{\partial\mu}{\partial\theta}\right)\left(r\frac{\partial\delta\sigma}{\partial r}+\delta\sigma+\frac{\partial\delta\mu}{\partial\theta}\right)\right.$$
$$\left.-p\left(r\frac{\partial\delta\sigma}{\partial r}+\delta\sigma+\frac{\partial\delta\mu}{\partial\theta}\right)\right]\mathrm{d}r\,\mathrm{d}\theta = 0. \tag{4.5}$$

Integrating by parts and setting the coefficients of $\delta\sigma$ and $\delta\mu$ in the resulting equations equal to zero we arrive at the partial differential equations

$$\begin{aligned}(\lambda+\nu)\frac{\partial}{\partial r}\left(\frac{\partial\sigma}{\partial r}+\frac{\sigma}{r}+\frac{1}{r}\frac{\partial\mu}{\partial\theta}\right)-\frac{\partial p}{\partial r} &= 0,\\ (\lambda+\nu)\frac{\partial}{\partial\theta}\left(\frac{\partial\sigma}{\partial r}+\frac{\sigma}{r}+\frac{1}{r}\frac{\partial\mu}{\partial\theta}\right)-\frac{\partial p}{\partial\theta} &= 0,\end{aligned} \tag{4.6}$$

which together imply, for some constant c, that

$$(\lambda + \nu)\left(\frac{\partial\sigma}{\partial r} + \frac{\sigma}{r} + \frac{1}{r}\frac{\partial\mu}{\partial\theta}\right) - p = c \tag{4.7}$$

in $\mathcal{R}_0$. We also have, from the integration by parts in the r direction, the boundary condition

$$\left[(\lambda + \nu)\left(\frac{\partial\sigma}{\partial r} + \frac{\sigma}{r} + \frac{1}{r}\frac{\partial\mu}{\partial\theta}\right) - p\right]\bigg|_{r=1} = 0, \tag{4.8}$$

which implies $c = 0$ in (4.7).

In polar coordinates the Cauchy-Riemann equations take the form

$$\frac{\partial\sigma}{\partial r} = \frac{\sigma}{r} + \frac{1}{r}\frac{\partial\mu}{\partial\theta}, \qquad \frac{\partial\mu}{\partial r} = \frac{\mu}{r} - \frac{1}{r}\frac{\partial\sigma}{\partial\theta}. \tag{4.9}$$

Substituting the first equation of (4.9) into (4.7) with $c = 0$ we obtain

$$\frac{\partial\sigma}{\partial r} = \frac{p}{2(\lambda + \nu)}, \tag{4.10}$$

which determines σ in $\mathcal{R}_0$ once σ has been given on the boundary $\mathcal{B}_0$. Assuming $\sigma = 0$ at the origin, the normal displacement component on the boundary is given by

$$\sigma(1,\theta) = \frac{1}{2(\lambda + \nu)}\int_0^1 p(r,\theta)\,\mathrm{d}r.$$

If we assume it is the modified region $\mathcal{R}$, with boundary $\mathcal{B}$, which is of direct interest, our problem becomes that of specifying $\sigma(1,\theta)$ such that the solution of this boundary component control problem yields a deformation $\mathcal{X}(X) = X + \Xi(X)$ mapping $\mathcal{R}_0$ to $\mathcal{R}$. In general this problem appears to be very difficult but we can obtain some results for $\mathcal{R}_0$ the unit disk presently under discussion. Given $\sigma(1,\theta)$, we assume it is achieved with the unique volume control $p(r,\theta)$ of least norm in $L^2(\mathcal{R}_0)$; then $p(r,\theta)$ is harmonic in that region. We expand the boundary values $p(1,\theta)$ in Fourier series to obtain

$$p(1,\theta) = \sum_{k=-\infty}^{\infty} p_k\,\mathrm{e}^{ik\theta}.$$

Then, as is well known, we have

$$p(r,\theta) = \sum_{k=-\infty}^{\infty} p_k\,r^{|k|}\,\mathrm{e}^{ik\theta} \tag{4.11}$$

in the whole disk. Assuming

$$\sigma(1,\theta) = \sum_{k=-\infty}^{\infty} \sigma_k\,\mathrm{e}^{ik\theta},$$

we then have, for $k = 0, 1, 2, \ldots$,

$$\sigma_k = \frac{p_k}{2(\lambda + \nu)}\int_0^1 r^{|k|}\,\mathrm{d}r = \frac{p_k}{2(\lambda + \nu)(|k| + 1)} \Rightarrow p_k = 2(\lambda+\nu)\,(|k|+1)\,\sigma_k. \tag{4.12}$$

From (4.12) we see that we can obtain $p(1,\theta) \in H^m(\mathcal{B}_0)$ if $\sigma(1,\theta) \in H^{m+1}([0,2\pi])$ for any nonnegative m. Then from the second of Eq. (4.9), we have ($\mu(\theta) = \mu(1,\theta)$)

$$\mu'(\theta) = \frac{p(1,\theta)}{2(\lambda+\nu)} - \sigma(1,\theta). \tag{4.13}$$

We can obtain $\sigma(r,\theta)$ from (4.10) as

$$\sigma(r,\theta) = \frac{1}{2(\lambda+\nu)}\int_0^r p(\rho,\theta)\,\mathrm{d}\rho = \sum_{k=-\infty}^{\infty} \frac{p_k\, r^{|k|+1}}{2(\lambda+\nu)(|k|+1)}\,\mathrm{e}^{ik\theta}. \tag{4.14}$$

Substitution of (4.10) into the first Eq. (4.9) followed by use of (4.11) and (4.14) gives a periodic function $\mu(\theta) \equiv \mu(1,\theta) \in H^{m+1}([0,2\pi])$:

$$\mu(r,\theta) = \sum_{|k|>0} \frac{p_k\, r^{|k|+1}}{2(\lambda+\nu)(|k|+1)}\,\frac{i}{k}\mathrm{e}^{ik\theta} - \sum_{|k|>0} p_k\, r^{|k|}\,\frac{i}{k}\mathrm{e}^{ik\theta}. \tag{4.15}$$

If we then substitute the formula for p_k in terms of σ_k appearing in (4.14) we obtain

$$\mu(r,\theta) = \sum_{|k|>0} \frac{2(\lambda+\nu)\,(|k|+1) - 1}{ik}\,\sigma_k\,\mathrm{e}^{ik\theta}. \tag{4.16}$$

Taking $r = 1$ we conclude that the map from $\sigma(1,\theta)$ to $\mu(1,\theta)$ is a bounded map from $H^m([0,2\pi])$ to itself for $m \geq 1$.

Let us suppose the target boundary $\mathcal{B}$ is described by an equation

$$r = 1+\rho(\theta), \quad 0 \leq \theta \leq 2\pi, \tag{4.17}$$

where $\rho(\theta)$ is of class C^2 with $\rho(0) = \rho(2\pi)$, $\rho'(0) = \rho'(2\pi)$. Then for each θ_0 the points

$$r = 0; \qquad r = 1+\sigma(1,\theta_0),\ \theta = \theta_0;$$
$$r = 1+\rho(\theta_0+\mu(1,\theta_0)),\ \theta = \theta_0+\mu(1,\theta_0)$$

must form a right triangle because the point $r = 1+\rho(\theta_0+\mu(1,\theta_0))$, $\theta = \theta_0+\mu(1,\theta_0)$ must lie on the line passing through $r = 1+\sigma(1,\theta_0)$, $\theta = \theta_0$ and orthogonal to the line joining the origin to this latter point. Thus

$$\frac{1+\sigma(1,\theta_0)}{1+\rho(\theta_0+\mu(1,\theta_0))} = \cos(\mu(1,\theta_0)).$$

Replacing θ_0 by the generic θ, we obtain the equation

$$\sigma(1,\theta) = \cos(\mu(1,\theta))\big(1 + \rho(\theta+\mu(1,\theta))\big) - 1. \tag{4.18}$$

Of course it is clear that if we start with $\rho(\theta)$, the normal displacement component $\sigma(1,\theta)$ and the tangential displacement component $\mu(1,\theta)$ will not be known at the outset. The problem is to determine those functions in such a way that the solution of the normal boundary component problem with specified $\sigma(1,\theta)$ will achieve the new boundary given by $r = \rho(\theta)$.

Theorem 4.1 *Let $\mathcal{R}_0$ be the unit disk in R^2 and let $\mathcal{R} \subset R^2$ be a region whose boundary, $\mathcal{B}$, is described by Eq.* (4.17) *with $\rho(\theta)$ as indicated there. Let*

$$a = \max_{[0,2\pi))} |\rho(\theta)|, \quad b = \max_{[0,2\pi))} \left|\rho'(\theta)\right|, \quad d = \max_{[0,2\pi))} \left|\rho''(\theta)\right|. \tag{4.19}$$

Then, if a, b, and d are sufficiently small positive numbers, there is a unique function $\sigma(\theta) \in H^1([0, 2\pi))$ such that the unique solution of the normal boundary component control problem with objective $\sigma(1, \theta) = \sigma(\theta)$, corresponding to a volume control $p(r, \theta)$ of least norm in $L^2([0, 2\pi))$, achieves a deformation $\mathcal{X}(X) = X + \Xi(X)$ mapping $\mathcal{R}_0$ onto $\mathcal{R}$, equivalently mapping $\mathcal{B}_0$, the unit circle in R^2, onto $\mathcal{B}$, the boundary of $\mathcal{R}$, described by $\rho(\theta)$.

Proof Let Σ_a be the metric space consisting of functions $\sigma(\theta) \in H^1([0, 2\pi))$ such that

$$\max_{[0,2\pi)} |\sigma(\theta)| \leq 2a, \qquad \left\|\sigma'\right\|_{L^2([0,2\pi))} \leq c,$$

where $c > 0$ is yet to be determined. The metric is that consistent with the norm, equivalent to the standard norm in $H^1([0, 2\pi))$,

$$\|\sigma\|_\Sigma = \max_{[0,2\pi)} |\sigma(\theta)| + \left\|\sigma'\right\|_{L^2([0,2\pi))}. \tag{4.20}$$

Given $\sigma(\theta) \in \Sigma$, we define $\mu(\theta)$ by the formula (4.16) with $r = 1$:

$$\mu(\theta) = \sum_{|k|>0} \frac{2(\lambda + \nu)(|k| + 1) - 1}{ik} \sigma_k e^{ik\theta}. \tag{4.21}$$

Then $\mu(\theta)$ lies in $H^1([0, 2\pi))$ and we have

$$\begin{aligned} \left\|\mu'\right\|_{L^2([0,2\pi))} &\leq 4(\lambda + \nu) \left\|\sigma'\right\|_{L^2([0,2\pi))} \leq 4(\lambda + \nu)c, \\ |\mu(\theta)| &\leq 4(\lambda + \nu)\sqrt{2\pi} \left\|\sigma'\right\|_{L^2([0,2\pi))} \leq 4(\lambda + \nu)\sqrt{2\pi}c. \end{aligned} \tag{4.22}$$

Then we define $\tilde{\sigma}(\theta)$ using (4.18), i.e.,

$$\tilde{\sigma}(\theta) = \cos(\mu(\theta)) \rho(\theta + \mu(\theta)) - \big(1 - \cos(\mu(\theta))\big). \tag{4.23}$$

Differentiating, we have

$$\begin{aligned} \tilde{\sigma}'(\theta) = &\cos(\mu(\theta))\rho'(\theta + \mu(\theta))(1 + \mu'(\theta)) \\ &- \sin(\mu(\theta))\mu'(\theta)\big(1 + \rho(\theta + \mu(\theta))\big). \end{aligned} \tag{4.24}$$

Now we want to estimate $\tilde{\sigma}$. Using (4.24) and (4.22) we have

$$\begin{aligned} \left\|\sigma'\right\|_{L^2([0,2\pi))} &\leq b\left(\sqrt{2\pi} + \left\|\mu'\right\|_{L^2([0,2\pi))}\right) + (1 + a)s \left\|\mu'\right\|_{L^2([0,2\pi))} \\ &\leq b\left(\sqrt{2\pi} + 4(\lambda + \nu)c\right) + 4s(1 + a)(\lambda + \nu)c, \end{aligned} \tag{4.25}$$

where $s = \max_{[0,2\pi)} |\sin(\mu(\theta))|$. We assume $c > 0$ chosen small enough so that, using the second estimate in (4.22), we have $4s(1 + a)(\lambda + \nu) \leq \frac{1}{2}$. Next we take $b > 0$ so that $b\left(\sqrt{2\pi} + 4(\lambda + \nu)\right) \leq \frac{1}{2}$. Then (4.25) gives

$$\left\|\tilde{\sigma}'\right\|_{L^2([0,2\pi))} \leq c.$$

Using (4.18) we have

$$|\tilde{\sigma}(\theta)| \leq a + \frac{|\mu(\theta)|^2}{2} \leq a + \frac{32c^2(\lambda + \nu)^2\pi}{2} \tag{4.26}$$

and, further restricting c as necessary, we have

$$|\tilde{\sigma}(\theta)| \leq 2a, \quad \theta \in [0, 2\pi). \tag{4.27}$$

Referring to (4.20), (4.26), and (4.27) together imply, for $\sigma \in \Sigma$, that $\tilde{\sigma} \in \Sigma$ as well.

Next we examine the contraction property. Since $\tilde{\sigma}$, from (4.23), depends only on ρ and μ, it is enough to compute

$$\frac{\partial \tilde{\sigma}}{\partial \mu}(\mu(\theta)) = -\sin(\mu(\theta))\rho(\theta + \mu(\theta)) + \cos(\mu(\theta))\rho'(\theta + \mu(\theta)) - \sin(\mu(\theta))$$

so that, with the definitions and restrictions above

$$\left|\frac{\partial \tilde{\sigma}}{\partial \mu}(\mu(\theta))\right| \leq s(a+1) + b.$$

In (4.24) σ' depends on both μ and μ' so we compute

$$\begin{aligned}\frac{\partial \tilde{\sigma}'}{\partial \mu}(\mu(\theta), \mu'(\theta)) = & -\sin(\mu(\theta))\rho'(\theta + \mu(\theta))(1 + \mu'(\theta)) \\ & + \cos(\mu(\theta))\rho''(\theta + \mu(\theta))(1 + \mu'(\theta)) \\ & - \cos(\mu(\theta))\mu'(\theta)(1 + \rho(\theta + \mu(\theta))) \\ & - \sin(\mu(\theta))\mu'(\theta)\rho'(\theta + \mu(\theta))\end{aligned}$$

from which, with d as in (4.19), we have the estimate (all norms in $L^2([0, 2\pi)$ unless indicated otherwise)

$$\begin{aligned}\left\|\frac{\partial \tilde{\sigma}'}{\partial \mu}(\mu(\theta), \mu'(\theta))\right\| \leq & (sb + d)\left(\sqrt{2\pi} + \|\mu'\|\right) \\ & + (1 + a)\|\mu'\| + sb\|\mu'\|.\end{aligned} \tag{4.28}$$

Further, we have

$$\frac{\partial \tilde{\sigma}'}{\partial \mu'}(\mu(\theta), \mu'(\theta)) = \cos(\mu(\theta))\rho'(\theta + \mu(\theta)) - \sin(\mu(\theta))(1 + \rho(\theta + \mu(\theta)),$$

leading to the estimate

$$\left\|\frac{\partial \tilde{\sigma}'}{\partial \mu'}(\mu(\theta), \mu'(\theta))\right\| \leq b + s(1 + a), \tag{4.29}$$

wherein the norm on the left is the norm of $\left(\partial \tilde{\sigma}' \big/ \partial \mu'\right)(\mu(\theta), \mu'(\theta))$ as a bounded linear operator on $L^2([0, 2\pi))$.

By restricting the size of d, and further restricting a, b, c, and s as necessary, the right-hand sides of the estimates (4.28) and (4.29) can be made as small as desired. Since we have seen in (4.22) that the operator carrying $\sigma(\theta)$, $\sigma'(\theta)$ to $\mu(\theta)$, $\mu'(\theta)$ is bounded relative to the norm (4.20), with our restrictions the operator carrying $\sigma(\theta)$, $\sigma'(\theta)$ is a contraction and the proof is complete.

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